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Biorthogonal Refinable Spline Functions

Tim N. T. Goodman

Abstract. We give a construction for refinable spline functions of degree n with compact support and simple knots in $\frac{1}{4}\mathbb{Z}$ which are biorthogonal to uniform B-splines of degree n with simple knots at $\frac{1}{3}\mathbb{Z}$.

§1. Introduction

A function is refinable if it is a linear combination of dilates of integer translates of itself. Such functions are central to multiresolution methods, in particular in the construction of wavelets. In general, refinable functions can be defined implicitly from the refinement equation which they satisfy, but explicit constructions of refinable functions are restricted mainly to spline functions, i.e. piecewise polynomials. If we require the natural condition that the integer translates of the univariate refinable spline function ϕ with compact support form a Riesz basis, then ϕ can only be a uniform B-spline with simple knots [4]. However there is more flexibility if we replace the single function ϕ by a refinable vector of spline functions $(\phi_1, \ldots, \phi_r)$. For a survey on refinable spline functions, see [2].

In multiresolution methods, orthogonality plays an important role. In [1], constructions are given for refinable functions whose integer translates are biorthogonal to a given refinable function ϕ, in particular when ϕ is a uniform B-spline with simple knots. However these dual functions are not defined explicitly. We give, in Section 3, constructions for refinable spline functions of compact support which are biorthogonal to uniform B-splines with simple knots. This requires refinable vectors of three functions: the uniform B-splines have knots in $\frac{1}{3}\mathbb{Z}$, while the dual functions have the same degree and simple knots in $\frac{1}{4}\mathbb{Z}$. The construction is based on a general result in Section 2 giving necessary and sufficient conditions for biorthogonality of certain vectors of (not necessarily refinable) functions in terms of a Grammian matrix.

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§2. Biorthogonal Basic Sets

Let $\phi_1,\ldots,\phi_r$ be compactly supported real-valued functions in $L^2(\mathbb{R})$. We say $\{\phi_1,\ldots,\phi_r\}$ is a basic set for a space V if V comprises all real, finite, linear combinations of integer translates of $\phi_1,\ldots,\phi_r$.

We say basic sets $\{\phi_1,\ldots,\phi_r\}$ and $\{\psi_1,\ldots,\psi_r\}$ are biorthogonal (or the basic set $\{\psi_1,\ldots,\psi_r\}$ is dual to $\{\phi_1,\ldots,\phi_r\}$) if

$$\int_{-\infty}^{\infty}\phi_i\psi_j(.-k)=\delta_{ij}\delta_{0k},\quad i,j=1,\ldots,r,\qquad k\in\mathbb{Z}.$$

A basic set $\{\phi_1,\ldots,\phi_r\}$ is said to be stable if $\{\phi_i(.-j):i=1,\ldots,r,j\in\mathbb{Z}\}$ forms a Riesz basis, i.e. for some A, $B>0$,

$$A\sum_{i=1}^{r}\sum_{j=-\infty}^{\infty}a_{ij}^2\le\int_{-\infty}^{\infty}[\sum_{i=1}^{r}\sum_{j=-\infty}^{\infty}a_{ij}\phi_i(.-j)]^2\le B\sum_{i=1}^{r}\sum_{j=-\infty}^{\infty}a_{ij}^2$$

for any $a_{ij}\in\mathbb{R}$, $i=1,\ldots,r$, $j\in\mathbb{Z}$.

It is shown in [3] that $\{\phi_1,\ldots,\phi_r\}$ is stable if and only if for each u in $\mathbb{R}$, there are integers $k_1,\ldots,k_r$ with

$$\det\left[\hat{\phi}_i(u+2\pi k_j)\right]_{i,j=1}^{r}\neq 0. \tag{1}$$

We shall say a matrix $M(z)=[M(z)_{ij}]_{i,j=1}^{r}$ of Laurent polynomials is invertible if it has an inverse which is a matrix of Laurent polynomials, i.e.

$$\det M(z)=az^l,\quad \text{some } a\neq 0,\quad l\in\mathbb{Z}.$$

Lemma 1. *If $\{\phi_1,\ldots,\phi_r\}$ is a stable basic set for V, then $\{\psi_1,\ldots,\psi_s\}$ in V also forms a stable basic set for V if and only if $s=r$ and*

$$\psi_i=\sum_{j=1}^{r}\sum_{k=-\infty}^{\infty}A_{ij}(k)\phi_j(.-k),\quad i=1,\ldots,r, \tag{2}$$

where the matrix of Laurent polynomials

$$A(z):=\left[\sum_{k=-\infty}^{\infty}A_{ij}(k)z^k\right]_{i,j=1}^{r}$$

is invertible.

Before proving this lemma, it will be useful to introduce the following vector notation. For a basic set $\{\phi_1,\ldots,\phi_r\}$, we let ϕ denote the column vector $(\phi_1,\ldots,\phi_r)^T$. Then we can write (2) as

$$\psi=\sum_{k=-\infty}^{\infty}A_k\phi(.-k), \tag{3}$$

where A_k is the matrix $[A_{ij}(k)]_{i,j=1}^{r}$. Taking Fourier transforms then gives

$$\hat{\psi}(u) = A(z)\hat{\phi}(u), \tag{4}$$

where $z = e^{-iu}$.

Proof of Lemma 1: Suppose that (2) holds, where $A(z)$ is invertible. From (4) we have

$$\hat{\phi}(u) = A(z)^{-1}\hat{\psi}(u).$$

Since $A(z)^{-1}$ is a matrix of Laurent polynomials, it follows that $\phi_1, \ldots, \phi_r$ are finite linear combinations of integer translates of $\psi_1, \ldots, \psi_r$. Since V comprises all finite linear combinations of integer translates of $\phi_1, \ldots, \phi_r$, it follows that $\{\psi_1, \ldots, \psi_r\}$ is a basic set for V.

Also for any u in $\mathbb{R}$, we may choose integers $k_1, \ldots, k_r$ so that (1) holds, and thus from (4),

$$\det[\hat{\psi}_i(u + 2\pi k_j)]_{i,j=1}^{r} = \det A(z) \det[\hat{\phi}_i(u + 2\pi k_j)]_{i,j=1}^{r} \neq 0.$$

So $\{\psi_1, \ldots, \psi_r\}$ is stable.

Conversely suppose that $\{\psi_1, \ldots, \psi_s\}$ is a stable basic set for V. Then there exist an $s \times r$ matrix $A(z)$ and an $r \times s$ matrix $B(z)$ of Laurent polynomials such that for $z = e^{-iu}$,

$$\hat{\psi}(u) = A(z)\hat{\phi}(u)$$

and

$$\hat{\phi}(u) = B(z)\hat{\psi}(u) = B(z)A(z)\hat{\phi}(u).$$

For any $u \in \mathbb{R}$ we may choose integers $k_1, \ldots, k_r$ so that (1) holds. Since

$$[\hat{\phi}_i(u + 2\pi k_j)]_{i,j=1}^{r} = B(z)A(z)[\hat{\phi}_i(u + 2\pi k_j)]_{i,j=1}^{r},$$

it follows that $B(z)A(z) = I_r$, the $r \times r$ identity matrix. Similarly $\hat{\psi}(u) = A(z)B(z)\hat{\psi}(u)$ and since $\{\psi_1, \ldots, \psi_s\}$ is stable, we can deduce as above that $A(z)B(z) = I_s$. Thus $s = r$ and $A(z)$ is invertible. □

Theorem 2. *Suppose that $\{\phi_1, \ldots, \phi_r\}$ and $\{\psi_1, \ldots, \psi_r\}$ are stable basic sets for V and W respectively. For $k \in \mathbb{Z}$, $i, j = 1, \ldots, r$, we define*

$$M_{ij}(k) := \int_{-\infty}^{\infty} \phi_i \psi_j(\cdot - k), \tag{5}$$

and let M denote the $r \times r$ matrix of Laurent polynomials given by

$$M(z) := \left[\sum_{k=-\infty}^{\infty} M_{ij}(k) z^k \right]_{i,j=1}^{r}. \tag{6}$$

Then there exist biorthogonal basic sets for V and W if and only if M is invertible. Moreover in this case, for any stable basic set for V there is a unique dual stable basic set for W.

Proof: Let $\{\tilde{\phi}_1, \ldots, \tilde{\phi}_r\}$ and $\{\tilde{\psi}_1, \ldots, \tilde{\psi}_r\}$ be any stable basic sets for V and W respectively. Then by Lemma 2.1 we have

$$\tilde{\phi} = \sum_{k=-\infty}^{\infty} A_k \phi(. - k), \quad \tilde{\psi} = \sum_{k=-\infty}^{\infty} B_k \psi(. - k), \tag{7}$$

where A_k, B_k are $r \times r$ matrices such that

$$A(z) := \sum_{k=-\infty}^{\infty} A_k z^k, \quad B(z) := \sum_{k=-\infty}^{\infty} B_k z^k, \tag{8}$$

are invertible.

For $k \in \mathbb{Z}$, $i, j = 1, \ldots, r$, define

$$\tilde{M}_{ij}(k) = \int_{-\infty}^{\infty} \tilde{\phi}_i \tilde{\psi}_j(. - k), \tag{9}$$

and let $\tilde{M}$ denote the $r \times r$ matrix of Laurent polynomials given by

$$\tilde{M}(z) := \left[\sum_{k=-\infty}^{\infty} \tilde{M}_{ij}(k) z^k \right]_{i,j=1}^{r}. \tag{10}$$

Then from (2.5)-(2.10) we have for $z = e^{-iu}$,

$$\tilde{M}(z) = A(z) M(z) B(z)^*, \tag{11}$$

where $B(z)^* = \overline{B(z)}^T = B(z^{-1})^T$.

Now by (2.9) and (2.10), $\tilde{\phi}$ and $\tilde{\psi}$ are biorthogonal if and only if $\tilde{M}(z) = I$. If this holds, then by (11), M must be invertible. Conversely, if M is invertible, then for any choice of $A(z)$ we can define $B(z)$ uniquely by

$$B(z) = (M(z)^{-1} A(z)^{-1})^*$$

so that (11) holds with $\tilde{M}(z) = I$. Thus if M is invertible, then for any stable basic set $\tilde{\phi}$ for V, there is a unique dual basic set $\tilde{\psi}$ for W. □

We remark that from the definition, any biorthogonal basic sets must have linearly independent integer translates. Thus if M as in Theorem 2.1 is invertible, any basic set for V or W which is stable must in fact have linearly independent integer translates.

Now for $m \in \mathbb{Z}$, $m \geq 2$, we say a space V of functions on $\mathbb{R}$ is m-refinable if

$$f \in V \Rightarrow f(\frac{\cdot}{m}) \in V.$$

If ϕ is a basic set for an m-refinable space V, then $\phi(\frac{\cdot}{m})$ is in V, and so

$$\phi(\frac{\cdot}{m}) = \sum_{k=-\infty}^{\infty} C_k \phi(. - k)$$

for some finite set of matrices C_k. Thus, ϕ satisfies the refinement equation

$$\phi = \sum_{k=-\infty}^{\infty} C_k \phi(m. - k).$$

In this paper we shall, for simplicity, consider only the case $m = 2$.

§3. Biorthogonal Refinable Splines

For any integer $n \geq 1$ we let N_n denote the uniform B-spline of degree n with simple knots at $0, 1, \ldots, n+1$. We now fix n, and define

$$\phi_1(x) = N_n(3x), \quad \phi_2(x) = N_n(3x-1), \quad \phi_3(x) = N_n(3x-2).$$

Then $\{\phi_1, \phi_2, \phi_3\}$ is a stable basic set for the space V of all spline functions of degree n with compact support and simple knots at $\frac{1}{3}\mathbb{Z}$. Clearly V is refinable. We wish to find a refinable space W of spline functions of compact support which has a basic set which is dual to $\{\phi_1, \phi_2, \phi_3\}$. From Theorem 2 we see that this is equivalent to finding a basic set $\{\psi_1, \psi_2, \psi_3\}$ for W such that the matrix M in (6) is invertible.

We shall choose

$$\psi_1(x) = N_n(2x), \quad \psi_2(x) = N_n(2x-1),$$

and ψ_3 to be a spline function of degree n with knots in $\frac{1}{4}\mathbb{Z}$ and support in $[\frac{1}{4}, 2n - \frac{1}{4}]$. Thus W is a space of spline functions of degree n and simple knots in $\frac{1}{4}\mathbb{Z}$, which contains all spline functions of degree n with compact support and simple knots in $\frac{1}{2}\mathbb{Z}$. For any function f in W, $f(\frac{\cdot}{2})$ has knots in $\frac{1}{2}\mathbb{Z}$ and so lies in W. Thus W is refinable.

Theorem 3. *We can choose ψ_3 as above so that there are biorthogonal basic sets for V and W, or equivalently that there is a unique basic set for W dual to the basic set $\{\phi_1, \phi_2, \phi_3\}$.*

Proof: The space W_0 of spline functions of degree n with simple knots in $\frac{1}{2}\mathbb{Z}$ and support in $[0, 2n]$ has dimension $3n$, while the space W_1 of spline functions of degree n with simple knots in $\frac{1}{4}\mathbb{Z}$ and support in $[0, 2n]$ has dimension $7n$. Thus we may choose linearly independent functions $f_1, \ldots, f_{4n}$ in W_1 with support in $[\frac{1}{4}, 2n - \frac{1}{4}]$ which together with W_0 span W_1. We write

$$\psi_3 = \sum_{k=1}^{4n} a_k f_k.$$

It remains to choose $a_1, \ldots, a_{4n}$ so that M in (6) is invertible. Now $M_{ij}(k) = 0$ except in the following cases: $M_{11}(k)$, $-\frac{n}{2} \le k \le \frac{n}{3}$; $M_{12}(k)$, $-\frac{n+1}{2} \le k \le \frac{n-1}{3}$; $M_{13}(k)$, $-2n+1 \le k \le \frac{n}{3}$; $M_{21}(k)$, $-\frac{n}{2} \le k \le \frac{n+1}{3}$; $M_{22}(k)$, $-\frac{n+1}{2} \le k \le \frac{n}{3}$; $M_{23}(k)$, $-2n+1 \le k \le \frac{n+1}{3}$; $M_{31}(k)$, $-\frac{n-1}{2} \le k \le \frac{n+2}{3}$; $M_{32}(k)$, $-\frac{n}{2} \le k \le \frac{n+1}{3}$; $M_{33}(k)$, $-2n+1 \le k \le \frac{n+2}{3}$. Then

$$\det M(z) = \sum_{k=-3n+1}^{n} b_k z^k,$$

for some numbers b_k, $-3n+1 \le k \le n$, which are linear functions of $a_1, \ldots, a_{4n}$. The condition $\det M(z) \equiv 1$ then gives $4n$ linear equations in $4n$ unknowns $a_1, \ldots, a_{4n}$ and we shall show that this system is non-singular.

Suppose, to the contrary, that the system is singular. Then we may choose $a_1, \ldots, a_{4n}$, not all zero, so that $\det M(z) \equiv 0$. Then the columns of M are linearly dependent in the sense that there are Laurent polynomials p_1, p_2, p_3, not all zero, so that

$$\sum_{j=1}^{3} M(z)_{ij} p_j(z) \equiv 0, \quad i = 1, 2, 3.$$

Writing

$$p_j(z) = \sum_{k=-\infty}^{\infty} c_j(k) z^k, \quad j = 1, 2, 3,$$

this becomes

$$\sum_{j=1}^{3} \sum_{k=-\infty}^{\infty} M_{ij}(k) z^k \sum_{l=-\infty}^{\infty} c_j(l) z^l \equiv 0, \quad i = 1, 2, 3,$$

i.e.

$$\sum_{j=1}^{3} \sum_{l=-\infty}^{\infty} M_{ij}(l) c_j(k-l) = 0, \quad i = 1, 2, 3, \quad k \in \mathbb{Z},$$

which on recalling (5) gives

$$\int_{-\infty}^{\infty} \phi_i(\cdot - k) f = 0, \quad i = 1, 2, 3, \quad k \in \mathbb{Z},$$

where

$$f = \sum_{j=1}^{3} \sum_{l=-\infty}^{\infty} c_j(-l) \psi_j(\cdot - l).$$

Thus f is orthogonal to V. Note that since the integer translates of ψ_1, ψ_2, ψ_3 are linearly independent, f is not identically zero. Since only a finite number of coefficients $c_j(-l)$, $j = 1, 2, 3$, $l \in \mathbb{Z}$, are non-zero, f has

compact support. Suppose that the support is $[\alpha, \beta + 2n]$, α, $\beta \in \mathbb{Z}$, but not in $[\alpha + 1, \beta + 2n]$ or $[\alpha, \beta + 2n - 1]$. It is easily seen that $\alpha \leq \beta$ and

$$f = \sum_{j=\alpha}^{\beta} c_j \psi_3(. - j) + g,$$

where $c_\alpha \neq 0 \neq c_\beta$ and g has support in $[\alpha, \beta + 2n]$ and lies in the space of splines of degree n with knots in $\frac{1}{2}\mathbb{Z}$, which we shall denote by Z.

We first note that for some h_1 in Z with support in $[\beta + \frac{3n}{2} - 1, \beta + 2n]$,

$$\eta_1 := c_\beta \psi_3(. - \beta) + h_1$$

is orthogonal to those elements of V with support in $[\beta + 2n - 1, \infty)$. Now

$$f - \frac{1}{c_\beta} \sum_{j=\alpha}^{\beta-1} c_j \eta_1(. - j + \beta)$$

is orthogonal to those elements of V with support in $[\beta + 2n - 2, \infty)$ and on this interval coincides with a function

$$\eta_2 := c_\beta \psi_3(. - \beta) + h_2,$$

where h_2 is in Z with support in $[\beta + \frac{3n}{2} - 2, \beta + 2n]$. Continuing in this way we recursively construct

$$\eta_j := c_\beta \psi_3(. - \beta) + h_j, \quad j = 1, \ldots, 4n,$$

which is orthogonal to those elements of V with support in $[\beta + 2n - j, \infty)$, where h_j is in Z with support in $[\beta + \frac{3n}{2} - j, \beta + 2n]$. In particular, η_{4n} has support in $[\beta - \frac{5n}{2}, \beta + 2n]$ and is orthogonal to those elements of V with support in $[\beta - 2n, \infty)$.

Choose F with $F^{(n+1)} = \eta_{4n}$ and with support in $(-\infty, \beta + 2n]$. Now for $j = 0, \ldots, 12n - 1$, let B_j be the B-spline of degree n with knots $\beta - 2n + \frac{j}{3}$, $\beta - 2n + \frac{j+1}{3}, \ldots, \beta - 2n + \frac{j+n+1}{3}$. Then since B_j is in V,

$$\begin{aligned} 0 = \int_{-\infty}^{\infty} B_j \eta_{4n} &= \int_{-\infty}^{\infty} B_j F^{n+1} \\ &= \left[\beta - 2n + \frac{j}{3}, \beta - 2n + \frac{j+1}{3}, \ldots, \beta - 2n + \frac{j+n+1}{3}\right] F. \end{aligned}$$

Thus F vanishes at $\beta - 2n + \frac{j}{3}$, $j = 0, \ldots, 12n - 1$. Now F coincides on $[\beta - 2n, \beta + 2n]$ with a spline G of degree $2n+1$ with support $[\beta - 3n - \frac{1}{2}, \beta + 2n]$ with knots at

$$\beta - 3n - \frac{1}{2}, \beta - 3n, \ldots, \beta - \frac{1}{2}, \beta, \beta + \frac{1}{4}, \beta + \frac{1}{2}, \ldots, \beta + 2n - \frac{1}{4}, \beta + 2n.$$

It then follows from the Schoenberg-Whitney Theorem [5] that G vanishes identically on $[\beta - 2n, \beta + 2n]$ and hence so does η_{4n}. So η_{4n} has support in $[\beta - \frac{5n}{2}, \beta - 2n]$ with knots in $\frac{1}{2}\mathbb{Z}$, and so η_{4n} vanishes identically, which is a contradiction. Thus the linear system is non-singular, which completes the proof. □

Finally we note that if ψ_3 is as in Theorem 3 and M is given by (6), then the basic set for W dual to $\{\phi_1, \phi_2, \phi_3\}$ is $\{\tilde{\psi}_1, \tilde{\psi}_2, \tilde{\psi}_3\}$ given by

$$\tilde{\psi} := \sum_{k=-\infty}^{\infty} C_k \psi(. - k),$$

where

$$\sum_{k=-\infty}^{\infty} C_k z^k = (M(z)^{-1})^* = (\text{adj } M(z))^* = (\text{adj } M(z^{-1}))^T.$$

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Tim N. T. Goodman
Dept. of Mathematics
The University
Dundee DD1 4HN, U.K.
tgoodman@mcs.dundee.ac.uk